Maximal (sequentially) compact topologies

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Dedicated to Professor Horst Herrlich on the occasion of his 65th birthday

Abstract

We revisit the known problem whether each compact topology is contained in a maximal compact topology and collect some partial answers to this question. For instance we show that each compact topology is contained in a compact topology in which convergent sequences have unique limits. We also answer a question of D.E. Cameron by showing that each sequentially compact topology is contained in a maximal sequentially compact topology. We finally observe that each sober compact T_1-topology is contained in a maximal compact topology and that each sober compact T_1-topology which is locally compact or sequential is the infimum of a family of maximal compact topologies.

1 Introduction

A topological space is called a *KC-space* (compare also [5]) provided that each compact set is closed. A topological space is called a *US-space* provided that each convergent sequence has a unique limit. It is known [19] that each Hausdorff space (= T_2-space) is a KC-space, each KC-space is a US-space and each US-space is a T_1-space (that is, singletons are closed); and no converse implication holds, but each first-countable US-space is a Hausdorff space.

A compact topology on a set X is called *maximal compact* provided that it is not strictly contained in a compact topology on X. It is known that a

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topological space is maximal compact if and only if it is a KC-space that is also compact [13]. These spaces will be called compact KC-spaces in the following.

Let us note that while there are many maximal compact topologies, minimal noncompact topologies do not exist: Any noncompact space X possesses a strictly increasing open cover $\{C_\alpha : \alpha < \delta\}$ of X where δ is a limit ordinal and C_0 can be assumed to be nonempty. Clearly then $\{\emptyset, X\} \cup \{C_\alpha : 0 < \alpha < \delta\}$ yields a base of a strictly coarser noncompact topology on X.

Maximal compact topologies need not be Hausdorff topologies [17] (see also [1, 15]). A standard example of a maximal compact topology that is not a Hausdorff topology is given by the one-point-compactification of the set of rationals equipped with its usual topology.

Indeed maximal compact spaces can be anti-Hausdorff (= irreducible), as we shall next observe by citing an example due to van Douwen (see [18]).

In order to discuss that example we first recall some pertinent definitions. A nonempty subspace S of a topological space is called *irreducible* (see e.g. [6]) if each pair of nonempty open sets of S intersects. Furthermore a topological space X called a *Fréchet space* (see [4, p. 53]) provided that for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a sequence of points of A converging to x. For the convenience of the reader we include a proof of the following observation (compare e.g. Math. Reviews 53#1519 of [14]).

Lemma 1 *Each Fréchet US-space X is a KC-space.*

Proof. Suppose that $x \in \overline{K}$ where K is a compact subspace of X. Because X is a Fréchet space, there is a sequence $(k_n)_{n \in \mathbf{N}}$ of points of K converging to x. Since K is compact, that sequence has a cluster point c in K. Because X is a Fréchet space, there is a subsequence of $(k_n)_{n \in \mathbf{N}}$ converging to c (compare [4, Exercise 1.6D]). Hence $x = c \in K$, because X is a US-space. We have shown that K is closed and conclude that X is a KC-space.

Example 1 *(van Douwen [18]) There exists a countably infinite compact Fréchet US-space that is anti-Hausdorff. By the preceding lemma that space is a KC-space and hence maximal compact. Thus there exists an infinite maximal compact space that is irreducible.*

On the other hand, by the result cited above each first-countable maximal compact (T_1-)topology satisfies the Hausdorff condition (compare [16, Theorem 8]).

2   Main problem and related questions

While it is known that each compact topology is contained in a compact T_1-topology (just take the supremum of the given topology with the cofinite topology) [16, Theorem 10], the question whether each compact topology is contained in a compact KC-topology (that is, is contained in a maximal compact topology) seems still to be open. Apparently that question was first asked by Cameron [3, p. 56, Question 5-1], but remained unanswered.

Of course, a simple application of Zorn's Lemma cannot help us here, since a chain of compact topologies need not have a compact supremum: Consider the sequence $(\tau_n)_{n \in \mathbf{N}}$ of topologies $\tau_n = \{\emptyset, \mathbf{N}\} \cup \{[1, k] : k \in \mathbf{N}, k \leq n\}$ $(n \in \mathbf{N})$ on the set $\mathbf{N}$ of positive integers.

On the other hand, for instance each infinite topological space X with a point x possessing only cofinite neighborhoods is clearly contained in a maximal compact topology: Just consider the one-point-compactification X_x of $X \setminus \{x\}$ where $X \setminus \{x\}$ is equipped with the discrete topology and x acts as the point at infinity.

The problem formulated above seems to be undecided even under additional strong conditions. Recall that a topological space is called *locally compact* provided that each of its points has a neighborhood base consisting of compact sets. Note that a locally compact KC-space is a regular Hausdorff space.

Problem 1 *Is each locally compact (resp. second-countable) compact topology contained in a maximal compact topology?*

The authors also do not know the answer to the following generalization of their main problem.

Problem 2 *Is each compact topology the continuous image of a maximal compact topology?*

In [16, Example 11] it is shown that a compact space need not be the continuous image of a compact T_2-space. In fact, a careful analysis of the argument reveals the following general fact (also stated in [7, 3.6]).

Proposition 1 *A KC-space Y that is the continuous image of a compact T_2-space X is a T_2-space.*

Proof. Let $f : X \to Y$ be a continuous map from a compact T_2-space onto a KC-space. Clearly f is a closed map, since f is continuous, X is compact and

Y is a KC-space. The conclusion follows, since obviously a closed continuous image of a compact T_2-space, is a T_2-space.

In this context also the following observation is of interest.

Proposition 2 *Let $f : X \rightarrow Y$ be a continuous map from a maximal compact space onto a topological space Y. Then Y is maximal compact if and only if the map f is closed.*

Proof. Suppose that $f : X \rightarrow Y$ is closed. Since $f^{-1}\{y\}$ is compact whenever $y \in Y$, we see that $f^{-1}K$ is compact whenever K is compact in Y (compare e.g. with the proof of [4, Theorem 3.7.2]). Since $f^{-1}K$ is closed, we conclude that $K = f(f^{-1}K)$ is closed and hence Y is a compact KC-space. For the converse, suppose that the map $f : X \rightarrow Y$ is not closed. Consequently there is a closed set F in X such that fF is not closed. Clearly the compact set fF witnesses the fact that Y is not a KC-space.

In connection with the preceding result we note (compare [2, Example 3.2]) that T_1-quotients of maximal compact spaces are not necessarily maximal compact.

Problem 3 *Are T_1-quotient topologies of maximal compact topologies contained in maximal compact topologies?*

Next we want to show that a weak version of our main problem has a positive answer.

Proposition 3 *Let (X, τ) be a compact T_1-space. Then there is a compact topology τ' finer than τ such that (X, τ') is a US-space.*

Proof. As usual two subsets A and B of X will be called *almost disjoint* provided that their intersection is finite. Let $\mathcal{M} = \{A_i : i \in I\}$ be a maximal (with respect to inclusion) family of pairwise almost disjoint injective sequences in X with a distinct τ-limit (that is, each $A_i \in \mathcal{M}$ is identified with $\{x_n : n \in \mathbf{N}\} \cup \{x\}$ where $(x_n)_{n \in \mathbf{N}}$ is an injective sequence in (X, τ) that converges to some point x different from each x_n). For each $i \in I$ and $m \in \mathbf{N}$, let $A_i^m = \{x_n : n \in \mathbf{N}, n \geq m\} \cup \{x\}$. Let τ' be the topology on X which is generated by the subbase $\tau \cup \{X \setminus A_i^m : i \in I, m \in \mathbf{N}\}$.

We first show that τ' is compact. Let $\mathcal{C}$ be a subcollection of $\mathcal{A}_\tau \cup \{A_i^n : i \in I, n \in \mathbf{N}\}$ with empty intersection. (Here, as in the following, $\mathcal{A}_\tau$ denotes the set of τ-closed sets.) Denote the intersection of $\mathcal{C}$ with $\mathcal{A}_\tau$ by $\mathcal{F}$. We want to show that there is a finite subcollection of $\mathcal{C}$ with an empty intersection. Of course,

it will be sufficient to find a finite subcollection of $\mathcal{C}$ with finite intersection. If $\mathcal{C} = \mathcal{F}$, then such a finite subcollection of $\mathcal{C}$ must exist by compactness of (X, τ). So in this case we are finished. If we have in our collection $\mathcal{C} \setminus \mathcal{F}$ two sets A_i^n and A_j^m with $i \neq j$, then their intersection will be finite. So in that case we are also done.

Therefore we can assume that the set $\mathcal{C} \setminus \mathcal{F}$ is nonempty and its elements are all of the form $A_{i_0}^m = \{x_n : n \in \mathbf{N}, n \geq m\} \cup \{a\}$ for some fixed $i_0 \in I$ and $n \in M$ where M is a nonempty subset of $\mathbf{N}$ and a is the chosen τ-limit of the sequence $(x_n)_{n \in \mathbf{N}}$.

If $a \in \cap \mathcal{F}$, then clearly $a \in \cap \mathcal{C}$ —a contradiction to $\cap \mathcal{C} = \emptyset$. So there is $F_0 \in \mathcal{F}$ such that $a \notin F_0$. Since F_0 is τ-closed and the injective sequence $(x_n)_{n \in \mathbf{N}}$ τ-converges to a, we conclude that $F_0 \cap \{x_n : n \in \mathbf{N}\}$ is finite, since otherwise $a \in F_0$. Hence for any $m \in M$ we have that $F_0 \cap A_{i_0}^m$ is finite and we are finished again.

We deduce from Alexander's subbase theorem that the topology τ' is compact.

Next we want to show that (X, τ') is a US-space. In order to reach a contradiction, suppose that there is some sequence $(x_n)_{n \in \mathbf{N}}$ that τ'-converges to x and y where x and y are distinct points in X. Replacing $(x_n)_{n \in \mathbf{N}}$ if necessary by a subsequence, we can and do assume that the sequence $(x_n)_{n \in \mathbf{N}}$ under consideration is injective and that x_n does not belong to $\{x, y\}$ whenever $n \in \mathbf{N}$. The claim just made is an immediate consequence of the fact that the original sequence $(x_n)_{n \in \mathbf{N}}$ attains each value at most finitely many often, since (X, τ) and thus (X, τ') is a T_1-space and $(x_n)_{n \in \mathbf{N}}$ has two distinct limits in (X, τ').

Then $(x_n)_{n \in \mathbf{N}}$ is an injective τ-convergent sequence having a τ-limit distinct from each x_n and by maximality of the collection $\mathcal{M}$ there is some $A_i = \{z_n : n \in \mathbf{N}\} \cup \{z\}$ where z denotes the chosen τ-limit of the sequence $(z_n)_{n \in \mathbf{N}}$) belonging to $\mathcal{M}$ such that $A_i \cap \{x_n : n \in \mathbf{N}\}$ has infinitely many elements. Suppose that there is some $p \in \mathbf{N}$ such that x or y does not belong to A_i^p. Then $X \setminus A_i^p$ is a τ'-open neighborhood of x or y, respectively, which does not contain infinitely many terms of the sequence $(x_n)_{n \in \mathbf{N}}$ which is impossible, because x and y are both τ'-limits of $(x_n)_{n \in \mathbf{N}}$. So there is no such $p \in \mathbf{N}$ and it necessarily follows that $x = z = y$ —a contradiction. We conclude that (X, τ') is a US-space.

Corollary 1 *Each compact topology is contained in a compact US-topology.*

Remark 1 *It is possible to strengthen the latter result further to the statement that each compact topology is contained in a compact topology with respect to which each compact countable set is closed.*

In order to see this we need the following two auxiliary results. We recall that a topological space is called *sequentially compact* provided that each of its sequences has a convergent subsequence.

Lemma 2 *Let X be a US-space and let $\{K_n : n \in \mathbf{N}\}$ be a countable family of sequentially compact sets in X having the finite intersection property. Then $\cap_{n \in \mathbf{N}} K_n$ is nonempty.*

Proof. For each $n \in \mathbf{N}$ find $x_n \in \cap_{i=1}^n K_i$. Then the sequence $(x_n)_{n \in \mathbf{N}}$ has a subsequence $(y_n)_{n \in \mathbf{N}}$ converging to $k \in K_1$, because K_1 is sequentially compact. Suppose that there is $m \in \mathbf{N}$ such that $k \notin K_m$. Since there is a tail of $(y_n)_{n \in \mathbf{N}}$ belonging to K_m and K_m is sequentially compact, there exists a subsequence of $(y_n)_{n \in \mathbf{N}}$ converging to some $p \in K_m$. Since X is a US-space, it follows that $k = p \in K_m$ —a contradiction. We conclude that $k \in \cap_{n \in \mathbf{N}} K_n$.

Lemma 3 *Each compact US-topology is contained in a compact topology with respect to which each compact countable set is closed.*

Proof. Let (X, τ) be a compact US-space and let τ' be the topology generated by the subbase $\tau \cup \{X \setminus K : K \subseteq X$ is countable and compact$\}$ on X.

We are going to show that τ' is compact. In order to reach a contradiction, assume that $\mathcal{C}$ is a subcollection of $\mathcal{A}_\tau \cup \{K \subseteq X : K$ is countable and compact$\}$ having the finite intersection property, but $\cap \mathcal{C} = \emptyset$. Since τ is compact, we deduce that some compact countable set K belongs to $\mathcal{C}$. Hence by countability of K there must exist a countable subcollection $\mathcal{D}$ of $\mathcal{C}$ such that $\cap \mathcal{D} = \emptyset$. Replace in $\mathcal{D}$ each member F of $\mathcal{D} \cap \mathcal{A}_\tau$ by its trace $F \cap K$ on K to get a countable collection $\mathcal{D}'$ of compact countable sets having the finite intersection property. By a result of [12], each compact countable space is sequentially compact and hence $\mathcal{D}'$ is a countable collection of sequentially compact sets in a US-space. Since $\cap \mathcal{D}' = \emptyset$, we have reached a contradiction to the preceding lemma. We conclude that τ' is compact by Alexander's subbase theorem. Evidently each compact countable set in (X, τ') is τ-compact and thus τ'-closed.

Problem 4 *Given some fixed cardinal $\kappa > \aleph_0$. Is each compact topology contained in a compact topology with respect to which each compact set of cardinality κ is closed?*

A modification of some of the arguments presented above allows us to answer positively the variant of the main problem (see [3, Question 8-1, p. 56]) formulated for sequential compactness instead of compactness.

Theorem 1 *Each sequentially compact topology τ on a set X is contained in a sequentially compact topology τ'' that is maximal among the sequential compact topologies on X.*

Proof. Since (X, τ) is sequentially compact and any convergent (sub)sequence has a constant or an injective subsequence, it is obvious that any sequence in (X, τ) has a subsequence that converges with respect to the supremum $\tau \vee \tau_c$ where τ_c denotes the cofinite topology on X. Therefore by replacing τ by $\tau \vee \tau_c$ if necessary, in the following we assume that the sequentially compact topology τ on X is a T_1-topology.

Define now a topology τ' on X in exactly the same way as above. We next show that (X, τ') is sequentially compact provided that (X, τ) is sequentially compact. Let $(y_n)_{n \in \mathbf{N}}$ be any sequence in X. It has a subsequence $(s_n)_{n \in \mathbf{N}}$ that converges to some point a in (X, τ), because (X, τ) is sequentially compact. If $(s_n)_{n \in \mathbf{N}}$ has a constant subsequence, then $(y_n)_{n \in \mathbf{N}}$ clearly has a convergent subsequence in (X, τ'). So by choosing an appropriate subsequence of $(s_n)_{n \in \mathbf{N}}$ if necessary, it suffices to consider the case that $(s_n)_{n \in \mathbf{N}}$ is injective and that $s_n \neq a$ whenever $n \in \mathbf{N}$. By maximality of $\mathcal{M}$ there is $A_i = \{z_n : n \in \mathbf{N}\} \cup \{z\}$ belonging to $\mathcal{M}$ such that $\{s_n : n \in \mathbf{N}\} \cap A_i$ is infinite. Hence there is a common injective subsequence of the injective sequences $(s_n)_{n \in N}$ and $(z_n)_{n \in \mathbf{N}}$ in this intersection. By definition of τ' that subsequence converges to z, because any basic τ'-neighborhood $G \cap \bigcap_{j=1}^{n} (X \setminus A_j^{k_j})$ of z where G is τ-open, $A_j \in \mathcal{M}$ and $k_j \in \mathbf{N}$ $(j = 1, \ldots, n)$ contains a tail of that subsequence, since $(z_n)_{n \in \mathbf{N}}$ τ-converges to z and $A_j \cap A_i$ is finite whenever $j = 1, \ldots, n$. We conclude that $(y_n)_{n \in \mathbf{N}}$ has a τ'-convergent subsequence and that (X, τ') is sequentially compact. As in the preceding proof, one argues that (X, τ') is a US-space.

We now define a new topology τ'' on X by declaring $A \subseteq X$ to be τ''-closed if and only if $x_n \in A$ whenever $n \in \mathbf{N}$ and $(x_n)_{n \in \mathbf{N}}$ converges to x in (X, τ') imply that $x \in A$. It is well-known and readily checked that τ'' is a topology finer than τ' on X with the property that any sequence $(x_n)_{n \in \mathbf{N}}$ that converges to x in (X, τ') also converges to x in (X, τ''). In particular, it follows that the space (X, τ'') is sequentially compact, because (X, τ') is sequentially compact.

Let K be a sequentially compact subset in (X, τ''). Suppose that $x_n \in K$ whenever $n \in \mathbf{N}$ and that the sequence $(x_n)_{n \in \mathbf{N}}$ converges to x in (X, τ'). Then there is a subsequence $(y_k)_{k \in \mathbf{N}}$ of $(x_n)_{n \in \mathbf{N}}$ that converges to $r \in K$ in (X, τ'), since K is sequentially compact in (X, τ'') and $\tau' \subseteq \tau''$. Thus $x = r$, since (X, τ') is a US-space and hence $x \in K$. By the definition of the topology τ'' we conclude that K is closed in (X, τ''). Therefore each sequentially compact subset of (X, τ'') is τ''-closed. By [2, Theorem 2.4] we conclude that τ'' is a

maximal sequentially compact topology on X, which is clearly finer than τ.

Let us finally mention another possibly even more challenging version of our main problem.

Problem 5 *Which (compact) T_1-topologies are the infimum of a family of maximal compact topologies?*

Evidently the cofinite topology on an infinite set X is the infimum of the family of maximal compact Hausdorff topologies of the one-point-compactifications X_x (where $x \in X$) that we have defined above. In Proposition 6 below we shall deal with a special answer to Problem 5.

3   Some further results

Let (X, τ) be a compact topological space. Denote by $\mathcal{A}_\tau$ (resp. $\mathcal{C}_\tau$) the set of all closed (resp. compact) sets of (X, τ).

Note that if τ and τ' are two compact topologies on a set X such that $\tau \subseteq \tau'$, then $\mathcal{A}_\tau \subseteq \mathcal{A}_{\tau'} \subseteq \mathcal{C}_{\tau'} \subseteq \mathcal{C}_\tau$. Of course, a topology τ is a compact KC-topology if and only if $\mathcal{A}_\tau = \mathcal{C}_\tau$.

As usual, a collection of subsets of X that is closed under finite intersections and finite unions will be called a *ring of sets* on X. We consider the set $\mathcal{M}_\tau$ of all rings $\mathcal{G}$ of sets ordered by set-theoretic inclusion on the topological space (X, τ) such that $\mathcal{A}_\tau \subseteq \mathcal{G} \subseteq \mathcal{C}_\tau$. Since $\mathcal{A}_\tau$ is such a ring, $\mathcal{M}_\tau$ is nonempty. If $\mathcal{K}$ is a nonempty chain in $\mathcal{M}_\tau$, then $\bigcup \mathcal{K}$ belongs to $\mathcal{M}_\tau$. By Zorn's lemma we conclude that $\mathcal{M}_\tau$ has maximal elements.

We shall call a collection $\mathcal{C}$ of subsets of a set X *compact** provided that each subcollection of $\mathcal{C}$ having the finite intersection property has nonempty intersection. We use this nonstandard convention in order to avoid any confusion with the concept of a compact topology.

Lemma 4 *Let (X, τ) be a compact topological space. If $\mathcal{G}$ is a maximal element in $\mathcal{M}_\tau$ that is a compact* collection, then $\mathcal{G} = \mathcal{A}_{\tau'}$ where τ' is a maximal compact topology finer than τ.*

Proof. Suppose that $\mathcal{G}$ is a maximal element in $\mathcal{M}_\tau$ that is compact*. Then $\{X \setminus K : K \in \mathcal{G}\}$ yields the subbase of a topology τ' on X. Observe that $\mathcal{A}_\tau \subseteq \mathcal{G} \subseteq \mathcal{A}_{\tau'}$. Since $\mathcal{G}$ is compact*, τ' will be compact, by Alexander's subbase theorem. Because τ' is compact, $\tau \subseteq \tau'$ implies that $\mathcal{A}_{\tau'} \subseteq \mathcal{C}_\tau$. Hence $\mathcal{A}_{\tau'}$ belongs to $\mathcal{M}_\tau$. We conclude that $\mathcal{G} = \mathcal{A}_{\tau'}$ by the maximality of $\mathcal{G}$.

It remains to be seen that τ' is maximal compact. If τ'' is a finer topology than τ' and compact, then $\mathcal{A}_{\tau'} \subseteq \mathcal{A}_{\tau''} \subseteq \mathcal{C}_\tau$. Hence by maximality of $\mathcal{G}$, $\mathcal{A}_{\tau''} = \mathcal{G} = \mathcal{A}_{\tau'}$ and so $\tau' = \tau''$. We have shown that τ' is maximal compact.

Proposition 4 *Let (X, τ) be a compact topological space such that each filterbase consisting of compact subsets has a nonempty intersection. Then τ is contained in a maximal compact topology τ'.*

Proof. Let $\mathcal{G}$ be any maximal element in $\mathcal{M}_\tau$ as defined above. Recall that $\mathcal{G}$ is closed under finite intersections. Hence any nonempty subcollection $\mathcal{G}'$ of $\mathcal{G}$ having the finite intersection property generates a filterbase consisting of compact sets on X. It follows from our hypothesis that $\mathcal{G}$ is a compact* collection. Furthermore by Lemma 4 we conclude that $\mathcal{G}$ is equal to the set of closed subsets of a maximal compact topology τ' that is finer than τ.

It is known and easy to see (compare [11, Theorem 6]) that if X is a compact KC-space, then the product X^2 is a KC-space if and only if X is a Hausdorff space. As an application of Proposition 4 we want to show however that the seemingly reasonable conjecture that the product topology of a large family of maximal compact topologies is no longer contained in a maximal compact topology is unfounded. In order to see this we next prove the following result.

Lemma 5 *Let $(X_i)_{i \in I}$ be a nonempty family of T_1-spaces such that each X_i has the property that every filterbase of compact sets has a nonempty intersection. Then the product $\Pi_{i \in I} X_i$ also has that property.*

Proof. We can (and do) assume that I is equal to some finite ordinal or an infinite limit ordinal ϵ. Let $\mathcal{F}$ be a filterbase of compact subsets on the product $\Pi_{\gamma < \epsilon} X_\gamma$.

For each $\alpha < \epsilon$ we shall inductively find $x_\alpha \in X_\alpha$ such that the set $A_\alpha = \{(y_\gamma)_{\gamma < \epsilon} \in \Pi_{\gamma < \epsilon} X_\gamma : y_\gamma = x_\gamma \text{ whenever } \gamma \leq \alpha\}$ satisfies $A_\alpha \cap K \neq \emptyset$ whenever $K \in \mathcal{F}$.

Suppose now that for some $\delta < \epsilon$ and all $\alpha < \delta$, $x_\alpha \in X_\alpha$ have been chosen such that $A_\alpha \cap K \neq \emptyset$ whenever $K \in \mathcal{F}$. Let us first establish the following claim.

Caim: $(\cap_{\alpha < \delta} A_\alpha) \cap K \neq \emptyset$ whenever $K \in \mathcal{F}$.

If δ is a successor ordinal, then by our induction hypothesis $A_{\delta - 1} \cap K \neq \emptyset$ whenever $K \in \mathcal{F}$. Therefore the claim is verified, since the family $\{A_\alpha : \alpha < \delta\}$ is monotonically decreasing. So let δ be a limit ordinal (possibly equal to 0) and fix $K \in \mathcal{F}$. Since for each $\alpha < \delta$, A_α is closed because every space X_α is a

T_1-space, and since $A_\alpha \cap K \neq \emptyset$ the claim holds by compactness of K and the monotonicity of the sequence $\{A_\alpha : \alpha < \delta\}$. (For the case that $\delta = 0$ as usual we use the convention that $\cap \emptyset = \Pi_{\gamma < \epsilon} X_\gamma$.)

Continuing now with the proof we next consider the filterbase $\{[\mathrm{pr}_{X_\delta}(\cap_{\alpha < \delta} A_\alpha \cap K)] : K \in \mathcal{F}\}$ of compact sets on X_δ. By our assumption on X_δ there exists some

$$x_\delta \in \cap_{K \in \mathcal{F}}[\mathrm{pr}_{X_\delta}(\cap_{\alpha < \delta} A_\alpha \cap K)].$$

It remains to show that for each $K \in \mathcal{F}$, $\{(y_\gamma)_{\gamma < \epsilon} \in \Pi_{\gamma < \epsilon} X_\gamma : y_\gamma = x_\gamma, \gamma \leq \delta\} \cap K \neq \emptyset$; but this is an immediate consequence of $x_\delta \in \mathrm{pr}_{X_\delta}(\cap_{\alpha < \delta} A_\alpha \cap K)$. Finally note that $\cap_{\alpha < \epsilon} A_\alpha = \{(x_\alpha)_{\alpha < \epsilon}\}$ and that — for ϵ exactly as in the case of the ordinal δ above — $\cap_{\alpha < \epsilon} A_\alpha \cap K \neq \emptyset$ whenever $K \in \mathcal{F}$. Hence the assertion of the lemma holds.

Proposition 5 *The product topology of a nonempty family of compact KC-topologies is contained in a maximal compact topology.*

Proof. Note first that in a compact KC-topology each filterbase of compact sets has a nonempty intersection. We conclude by the preceding lemma and Proposition 4 that the compact product topology of an arbitrary nonempty family of maximal compact topologies is contained in a maximal compact topology.

Corollary 2 *Let $(X_i)_{i \in I}$ be a nonempty family of spaces each of which is contained in a maximal compact topology. Then also their product topology is contained in a maximal compact topology.*

4   Sobriety and maximal compactness

Note that the closure of each irreducible subspace of a topological space is irreducible. Recall also that a topological space is called *sober* (see e.g. [6]) provided that every irreducible closed set is the closure of some unique singleton. Clearly each Hausdorff space is sober. Furthermore a subset of a topological space is called *saturated* provided that it is equal to the intersection of its open supersets.

A short proof of the following result is given in [8].

Let $\{K_i : i \in I\}$ be a filterbase of (nonempty) compact saturated subsets of a sober space X. Then $\bigcap_{i \in I} K_i$ is nonempty, compact, and saturated, too; and an open set U contains $\bigcap_{i \in I} K_i$ iff U contains K_i for some $i \in I$.

Corollary 3 *Let (X, τ) be a compact sober T_1-space. Then τ is contained in some maximal compact topology τ'.*

Proof. Since all (compact) sets in a T_1-space are saturated, the condition stated in Proposition 4 is satisfied by the result just cited. The statement then follows from Proposition 4.

Problem 6 *Characterize those sober compact topologies that are contained in a maximal compact topology.*

Remark 2 Let us observe that the maximal compact topology τ' obtained in Corollary 3 will be sober, since the only irreducible sets with respect to the coarser topology τ are the singletons. Van Douwen's example [18] mentioned earlier shows that a maximal compact topology need not be (contained in) a compact sober topology.

Example 2 Note that the closed irreducible subsets of the one-point-compactification X (of the Hausdorff space) of the rationals are the singletons: Any finite subset of a T_1-space with at least two points is discrete and hence not irreducible. Moreover any infinite subset of X contains two distinct rationals and thus cannot be irreducible. We conclude that an arbitrary power of X is a compact, sober T_1-space, because products of sober spaces are sober (see e.g. [6, Theorem 1.4]).

In the light of the proof of Proposition 4 one wonders which compact sober T_1-topologies can be represented as the infimum of a family of maximal compact topologies (compare Problem 5). Our next result provides a partial answer to this question. An interesting space satisfying the hypothesis of Proposition 6 is a T_1-space constructed in [10]: It has infinitely many isolated points although each open set is the intersection of two compact open sets. (It was noted in the discussion [10, p. 212] that that space is locally compact and sober.)

Recall that a topological space X is called *sequential* (see [4, p. 53]) provided that a set $A \subseteq X$ is closed if and only if together with any sequence it contains all its limits in X.

Proposition 6 *Each compact sober T_1-space (X, τ) which is locally compact or sequential is the infimum of a family of maximal compact topologies.*

Proof. Note that if K belongs to the closed sets of a maximal compact topology σ finer than τ, then K is compact with respect to σ and thus with respect to τ. In order to verify the statement, it therefore suffices to construct

for any compact set C that is not closed in (X, τ) a maximal compact topology σ finer than τ in which C is not closed.

So let C be a compact set that is not closed in (X, τ). In (X, τ) we shall next find a compact set K_0 such that $K_0 \cap C$ is not compact.

Suppose first that X is locally compact.

Then there is $x \in X$ such that $x \in \mathrm{cl}_\tau C \setminus C$. Let $\mathcal{F} = \{K : K$ is a compact neighborhood at x in $(X, \tau)\}$. Of course, $\cap \mathcal{F} = \{x\}$, since X is a locally compact T_1-space. Suppose that $K \cap C$ is compact in (X, τ) whenever $K \in \mathcal{F}$.

Then $\{K \cap C : K \in \mathcal{F}\}$ is a filterbase of compact saturated sets in X. According to the result cited above from [8], we have $\cap \mathcal{F} \cap C = \bigcap \{K \cap C : K \in \mathcal{F}\} \neq \emptyset$. Since $x \notin C$, we have reached a contradiction. Thus there is a compact neighborhood K_0 of x such that $K_0 \cap C$ is not compact in (X, τ).

Suppose next that X is sequential. Since C is not closed, there is a sequence $(x_n)_{n \in \mathbf{N}}$ in C converging to some point $x \in X$ such that x does not belong to C. Assume that $\{(\{x\} \cup \{x_n : n \in \mathbf{N}, n \geq m\}) \cap C : m \in \mathbf{N}\}$ is a filterbase of compact sets. Clearly its intersection is empty, because τ is a T_1-topology and $(x_n)_{n \in \mathbf{N}}$ converges to x—a contradiction. Hence there is $m \in \mathbf{N}$ such that $(\{x\} \cup \{x_n : n \in \mathbf{N}, n \geq m\}) \cap C$ is not compact. Denote the compact set $\{x\} \cup \{x_n : n \in \mathbf{N}, n \geq m\}$ by K_0.

So our claim holds in either case.

Note now that $\tau \cup \{X \setminus K_0\}$ is a subbase for a compact topology τ' on X that is also sober and T_1. By Corollary 3 there is a maximal compact topology τ'' finer than τ'. Observe that $X \setminus C \notin \tau''$: Otherwise $C \in \mathcal{A}_{\tau''}$ and, since $K_0 \in \mathcal{A}_{\tau''}$, also $K_0 \cap C \in \mathcal{A}_{\tau''}$. Therefore $K_0 \cap C \in \mathcal{C}_{\tau''}$ and $K_0 \cap C \in \mathcal{C}_\tau$ —a contradiction. Thus indeed $X \setminus C \notin \tau''$. We conclude that τ is the infimum of a family of maximal compact topologies.

Observe that the argument above also yields the following results.

Corollary 4 *Each locally compact sober T_1-space in which the intersection of any two compact sets is compact is a KC-space (and therefore is a regular Hausdorff space).*

Corollary 5 *Each sequential sober T_1-space in which the intersection of any two compact sets is compact is a KC-space.*

We next give an example of a compact sober T_1-topology that is not the infimum of a family of maximal compact topologies.

Example 3 Let Y be an uncountable set and let $-\infty$ and ∞ be two distinct points not in Y. Set $X = Y \cup \{-\infty, \infty\}$. Each point of Y is supposed

to be isolated. The neighborhoods of ∞ are the cofinite sets containing ∞ and the neighborhoods of $-\infty$ are the cocountable sets containing $-\infty$. Clearly X is a compact sober T_1-space.

Next we show that with respect to the defined topology τ a subset A of X is compact and not closed if and only if A is uncountable, $\infty \in A$ and $-\infty \notin A$: Indeed, if $\infty \in A$, then A is clearly compact and if A is uncountable and $-\infty \notin A$, then A cannot be closed. In order to prove the converse suppose that A is compact and not closed in (X, τ). Then A is certainly infinite. It therefore follows from compactness of A that $\infty \in A$. Since A is not closed, we conclude that $-\infty \in \overline{A} \setminus A$ and hence A is uncountable.

Of course, if τ' is a maximal compact topology such that $\tau \subseteq \tau'$, then $\mathcal{A}_\tau \subseteq \mathcal{A}_{\tau'} \subseteq \mathcal{C}_{\tau'} \subseteq \mathcal{C}_\tau$. Observe that the topology τ'' generated by the subbase $\{\{-\infty\}\} \cup \tau$ clearly yields a compact T_2-topology finer than τ. Obviously, $\mathcal{C}_\tau \setminus \mathcal{A}_\tau \subseteq \mathcal{A}_{\tau''}$ by the description found above of the nonclosed compact sets in (X, τ). Thus $\mathcal{A}_{\tau''} = \mathcal{C}_\tau$. We conclude that τ'' is finer than any maximal compact topology containing τ. Hence τ'' is the only maximal compact topology (strictly) finer than τ.

Let us recall that a topological space is called *strongly sober* provided that the set of limits of each ultrafilter is equal to the closure of some unique singleton. Of course, each compact Hausdorff space satisfies this condition.

We finally observe that each locally compact strongly sober topological space (X, τ) possesses a finer compact Hausdorff topology; just take the supremum of τ and its dual topology (see e.g. [9, Theorem 4.11]). By definition, the latter topology is generated by the subbase $\{X \setminus K : K$ is compact and saturated in $X\}$ on X.

No characterization seems to be known of those topologies that possess a finer compact Hausdorff topology.

References

[1] V.K. Balachandran, *Minimal bicompact space*, J. Ind. Math. Soc. (N.S.) **12** (1948), 47–48.

[2] D.E. Cameron, *Maximal and minimal topologies*, Trans. Amer. Math. Soc. **160** (1971), 229–248.

[3] D.E. Cameron, *A survey of maximal topological spaces*, Topology Proc. **2** (1977), 11–60.

[4] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[5] W.G. Fleissner, *A T_B-space which is not Katetov T_B*, Rocky Mountain J. Math. **10** (1980), 661–663.

[6] R.-E. Hoffmann, *On the sobrification remainder ${}^s X - X$*, Pacific J. Math. **83** (1979), 145–156.

[7] R.-E. Hoffmann, *On weak Hausdorff spaces*, Arch. Math. (Basel) **32** (1979), 487–504.

[8] K. Keimel and J. Paseka, *A direct proof of the Hofmann-Mislove theorem*, Proc. Amer. Math. Soc. **120** (1994), 301–303.

[9] R. Kopperman, *Asymmetry and duality in topology*, Topology Appl. **66** (1995), 1–39.

[10] H.-P.A. Künzi and S. Watson, *A nontrivial T_1-space admitting a unique quasi-proximity*, Glasgow Math. J. **38** (1996), 207–213.

[11] N. Levine, *When are compact and closed equivalent?*, Amer. Math. Monthly **72** (1965), 41–44.

[12] N. Levine, *On compactness and sequential compactness*, Proc. Amer. Math. Soc. **54** (1976), 401–402.

[13] A. Ramanathan, *Minimal-bicompact spaces*, J. Ind. Math. Soc. (N.S.) **12** (1948), 40–46.

[14] M.J. Reed, *Hausdorff-like separation properties and generalizations of the first countability axiom*, Tamkang J. Math. **5** (1974), 197–201.

[15] N. Smythe and C.A. Wilkins, *Minimal Hausdorff and maximal compact spaces*, J. Austral. Math. Soc. **3** (1963), 167–171.

[16] A.H. Stone, *Compact and compact Hausdorff*, in: Aspects of Topology, pp. 315–324, London Math. Soc., Lecture Note Ser. **93**, Cambridge Univ. Press, Cambridge, 1985.

[17] H. Tong, *Note on minimal bicompact spaces (preliminary report)*, Bull. Amer. Math. Soc. **54** (1948), 478–479.

[18] E.K. van Douwen, *An anti-Hausdorff Fréchet space in which convergent sequences have unique limits*, Topology Appl. **51** (1993), 147–158.

[19] A. Wilansky, *Between T_1 and T_2*, Amer. Math. Monthly **74** (1967), 261–266.

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